ECEN 214 506-511

Electrical Circuit Theory
Exam #3 Review: Chapters 8-9

SPRING 2026

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Announcements

- Exam 3 – Wednesday, December 3
- Exam 3 Review – will be posted in Canvas. Covers chapters 8-10. You should know the theory that we cover through Monday on Chapter 10. However, I won't have a problem from Chapter 10.
- Exam 3 – Formula sheet. It is posted on Canvas for your review and as a study tool. This is the same formula sheet that will be used for the Final exam.
- I will provide a formula sheet on Exam day.
- You should focus your studying on Textbook Examples and Homework Problems
- Old Exam – An old exam is posted in Canvas to see the format and provide additional review problems

8.1 RLC Circuits

Figure 8.1 Natural response of a parallel RLC circuit

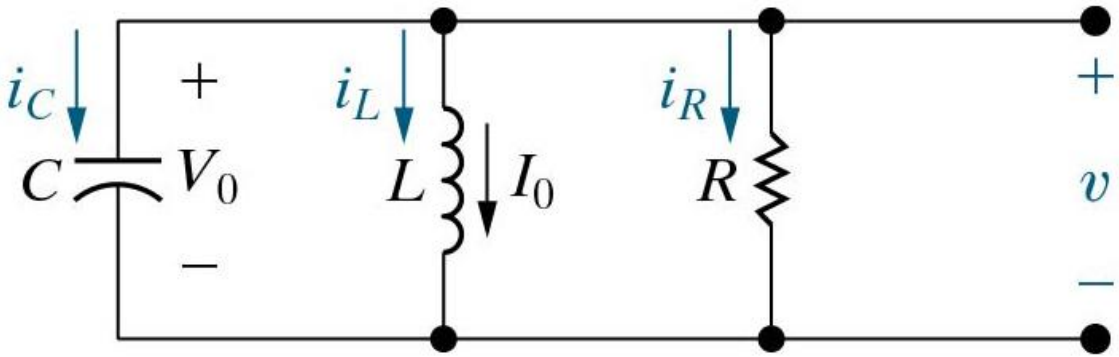


Figure 8.2 Step response of a parallel RLC circuit

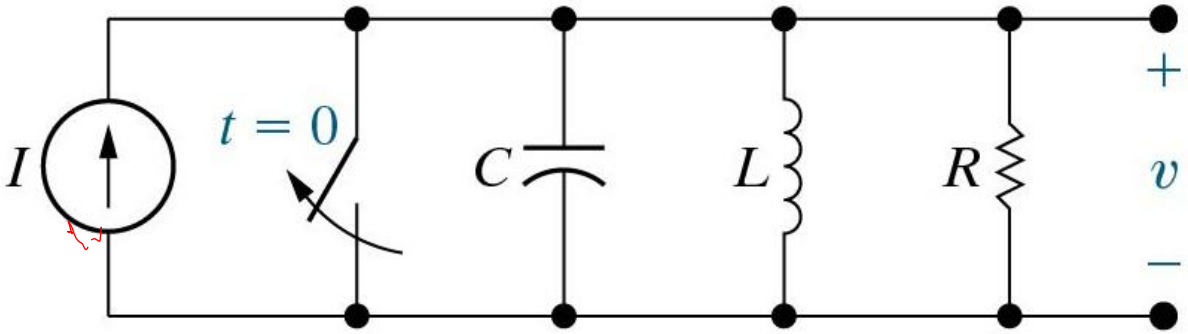


Figure 8.3 Natural response of a series RLC circuit

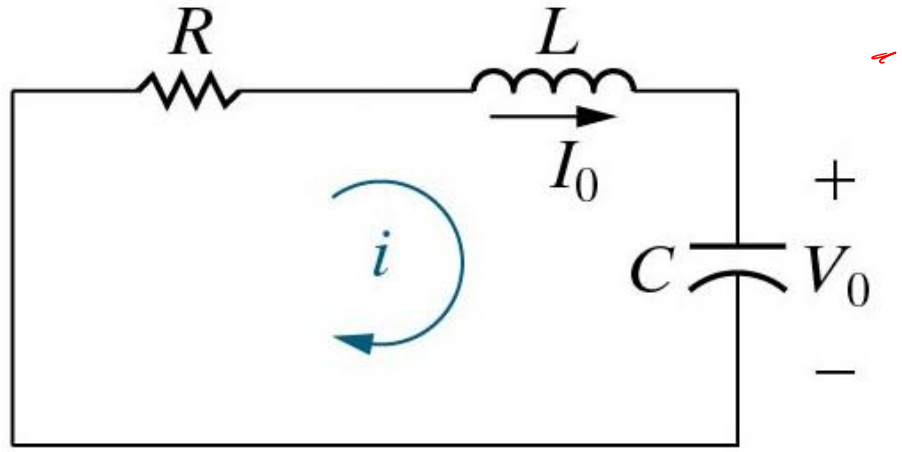
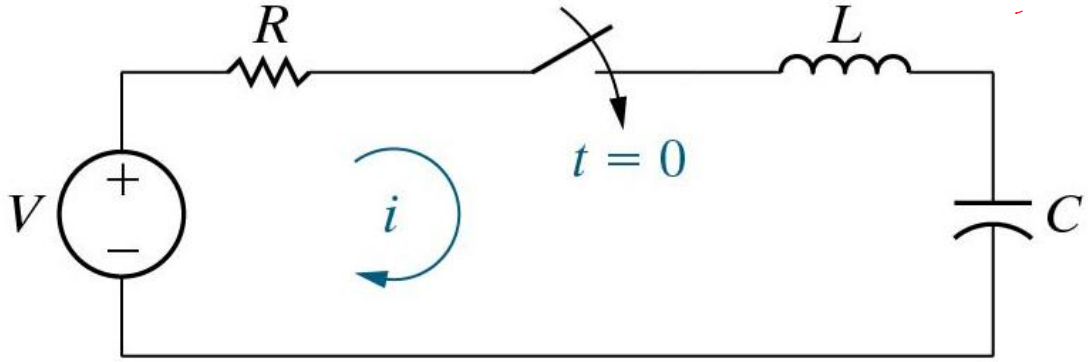
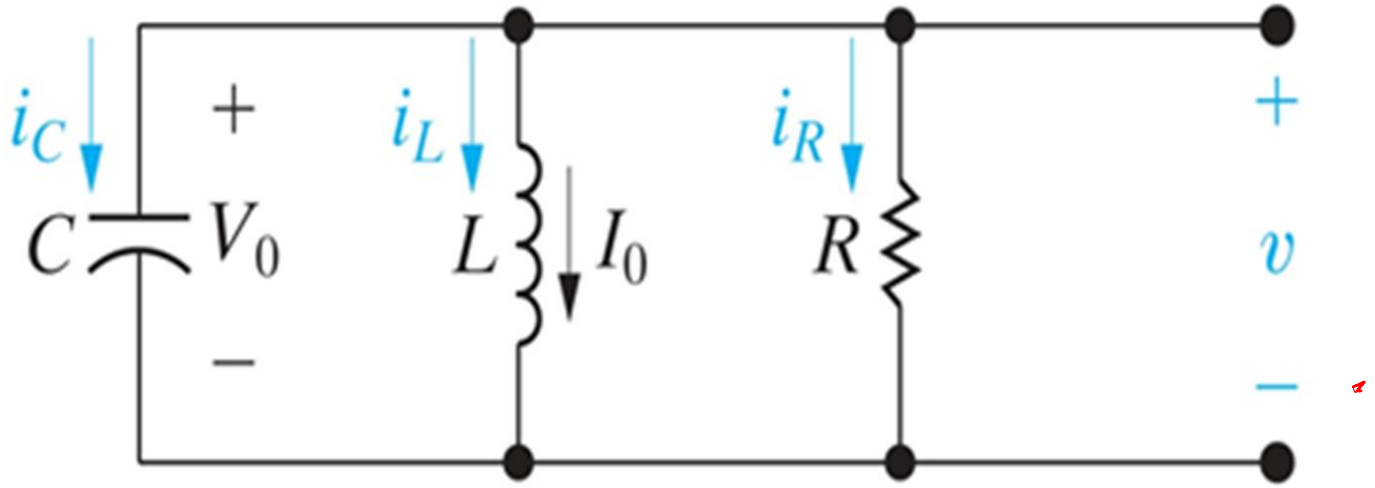


Figure 8.4 Step response of a series RLC circuit

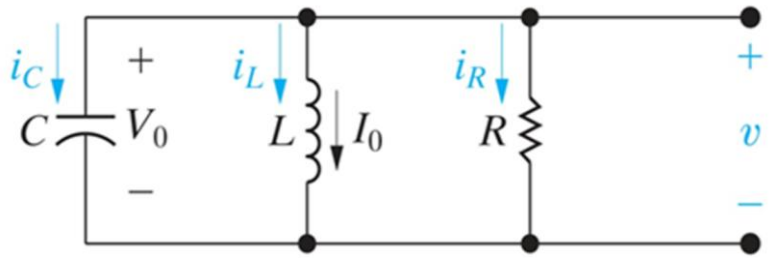


8.1 Natural Response of Parallel RLC Circuits

- The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t* ≥ *0*.



8.2 The Forms of the Natural Response of a Parallel RLC Circuit -- Summary of Equations



Describing equation:

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

TABLE 8.2 Equations for analyzing the natural response of parallel *RLC* circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1e^{s_1t} + A_2e^{s_2t}, \text{ t} \geq 0$
	$v(0^+) = A_1 + A_2 = V_0$
	$\frac{dv(0^+)}{dt} = s_1A_1 + s_2A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t, \text{ t} \geq 0$
	$v(0^+) = B_1 = V_0$
	$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t}, \text{ t} \geq 0$
	$v(0^+) = D_2 = V_0$
	$\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

8.2 The Forms of the Natural Response of a Parallel RLC Circuit Analysis Methods

1. Determine the initial capacitor voltage (V_0) and inductor current (I_0) from the circuit.
2. Determine the values of α and ω_0 using the equations in Table 8.2.
3. If $\propto^2 > \omega_0^2$, the response is overdamped, $\boldsymbol{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$, $t \geq 0$;

If $\propto^2 < \omega_0^2$, the response is underdamped, $\boldsymbol{v(t) = B_1 e^{-\alpha t} cos\omega_d t + B_2 e^{-\alpha t} sin\omega_d t}$, $t \geq 0$;

If $\propto^2 = \omega_0^2$, the response is critically damped, $\boldsymbol{v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$, $t \geq 0$;

4. If the response is overdamped, calculate s_1 and s_2 using the equations in table 8.2.

If the response is underdamped calculate ω_d using the equation in table 8.2.

5. If the response is overdamped, calculate A_1 and A_2 by simultaneously solving the equations in table 8.2.

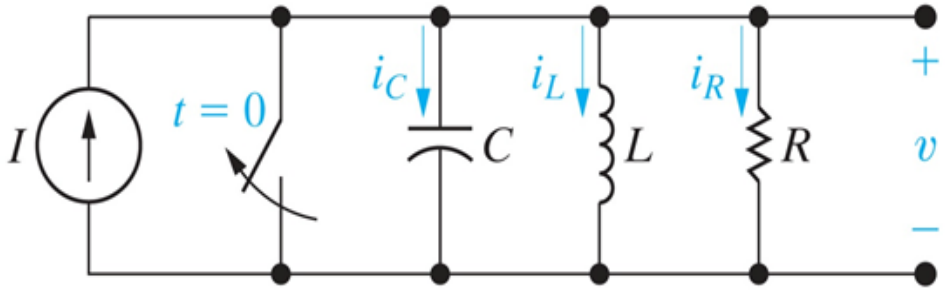
If the response is underdamped, calculate B_1 and B_2 by simultaneously solving the equations in table 8.2

If the response is critically damped, calculate D_1 and D_2 by simultaneously solving the equations in table 8.2

6. Write the equation for **v(t)** from Step 3 using the results of Steps 4 and 5; find any desired branch currents.

8.3 Step Response of Parallel RLC Circuits

- The problem – find the response for $t \geq 0$. Note that there may or may not be initial energy stored in the inductor and capacitor!



8.3 Step Response of Parallel RLC Circuits

- Describing equation: $\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$
- This equation is
  - Second order
  - Non-homogeneous
  - Ordinary differential equation
  - With constant coefficients
- Characteristics equation: $s^2 + (1/RC)s + (1/LC) = 1/LC$
- **Same characteristics equation as that of the natural response!**
- Therefore, $i_L(t) = I_F + I_N$,
- where I_F **is the particular solution (steady state/forced/final value)** and
- I_N **(transient) is the homogeneous solution** which has the same form as the natural solution and like before can be categorized into overdamped, underdamped and critically damped depending on the values of R, L and C.

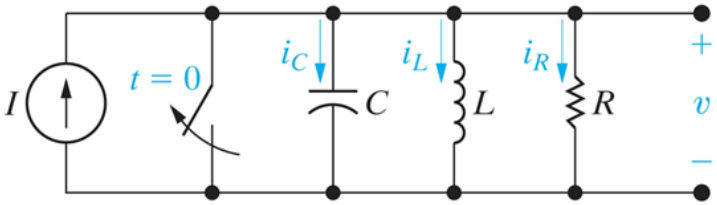
8.3 Summary of Equations for Analyzing the Step Response of Parallel RLC Circuits

TABLE 8.3 Equations for analyzing the step response of parallel *RLC* circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = \frac{I}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i_L(t) = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}, \quad t \geq 0$
	$i_L(0^+) = I_f + A_1' + A_2' = I_0$
	$\frac{di_L(0^+)}{dt} = s_1 A_1' + s_2 A_2' = \frac{V_0}{L}$
$\alpha^2 < \omega_0^2$: underdamped	$i_L(t) = I_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t, \quad t \geq 0$
	$i_L(0^+) = I_f + B_1' = I_0$
	$\frac{di_L(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = \frac{V_0}{L}$
$\alpha^2 = \omega_0^2$: critically damped	$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, \quad t \geq 0$
	$i_L(0^+) = I_f + D_2' = I_0$
	$\frac{di_L(0^+)}{dt} = D_1' - \alpha D_2' = \frac{V_0}{L}$

Describing equation:

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$



8.3 Step Response of a Parallel RLC Circuit

1. Determine the initial capacitor voltage (V_0), the initial inductor current (I_0), and the final inductor current (I_f) from the circuit.
2. Determine the values of α and ω_0 using the equations in Table 8.3.
3. If $\propto^2 > \omega_0^2$, the response is overdamped, $\boldsymbol{i(t) = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}}$, $t \geq 0^+$;

If $\propto^2 < \omega_0^2$, the response is underdamped, $\boldsymbol{i(t) = I_f + B_1' e^{-\alpha t} cos\omega_d t + B_2' e^{-\alpha t} sin\omega_d t}$, $t \geq 0^+$;

If $\propto^2 = \omega_0^2$, the response is critically damped, $\boldsymbol{i(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}}$, $t \geq 0^+$;

4. If the response is overdamped, calculate s_1 and s_2 using the equations in table 8.3.

If the response is underdamped calculate ω_d using the equation in table 8.3.

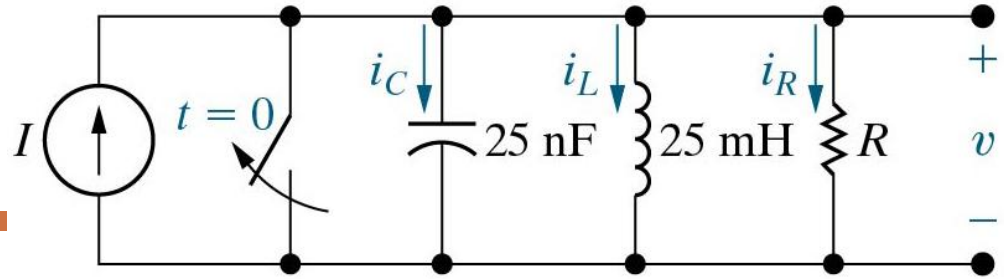
5. If the response is overdamped, calculate A_1' and A_2' by simultaneously solving the equations in table 8.3.

If the response is underdamped, calculate B_1' and B_2' by simultaneously solving the equations in table 8.3.

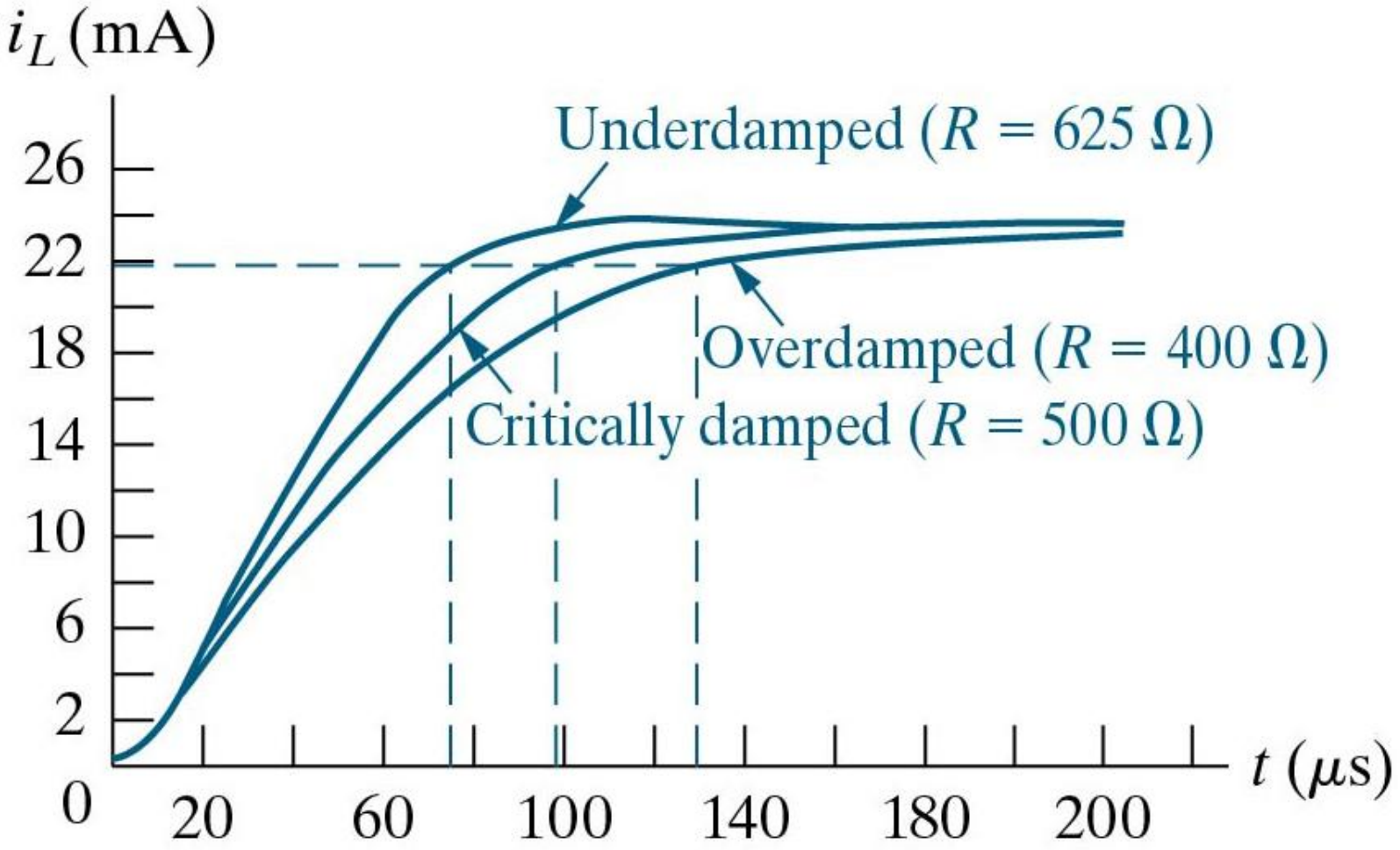
If the response is critically damped, calculate D_1' and D_2' by simultaneously solving the equations in table 8.3.

6. Write the equation for $i_L(t)$ from Step 3 using the results of Steps 4 and 5; find the inductor voltage and any desired branch currents.

Fig 8.14 – Current plots for three step response forms for parallel RLC circuit in Ex. 8.6-8.8

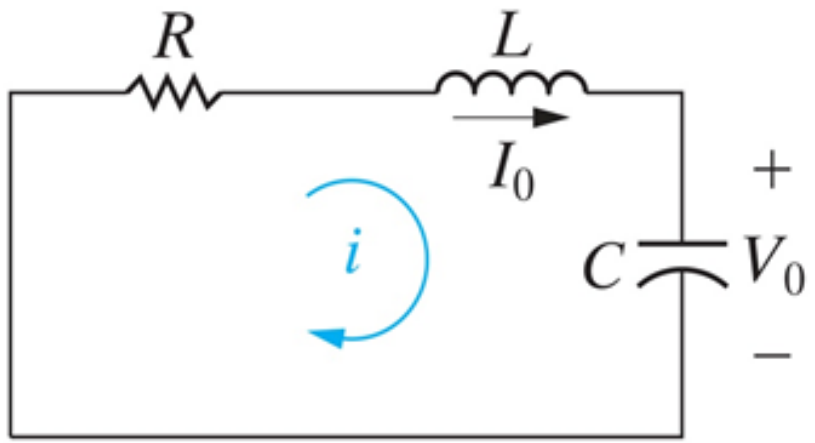


$I_f = 24\ mA$

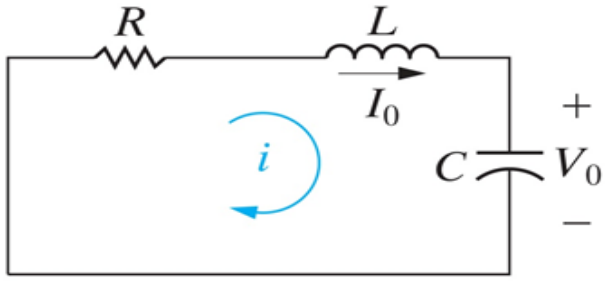


8.4 Natural and Step Responses of Series RLC Circuits

- The problem – given initial energy stored in the inductor and/or capacitor, find *i(t)* for $t \geq 0$.



8.4 Natural and Step Responses of Series RLC Circuits



- The Natural Response problem – given initial energy stored in the inductor and/or capacitor, find *i(t)* for $t \geq 0$.
- Characteristic Equation
  - $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$ (8.28)
  - $s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$
  - $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

    where $\alpha = \frac{R}{2L}$ (the neper frequency in rad/s) (8.29)

    and $\omega_0 = \frac{1}{\sqrt{LC}}$ (the resonant radian frequency in rad/s)

- Notice the formula for α is different here compared to the parallel RLC circuit!
- Just like in the parallel RLC circuit, $\alpha^2 - \omega_0^2$ will determine the nature of the response.

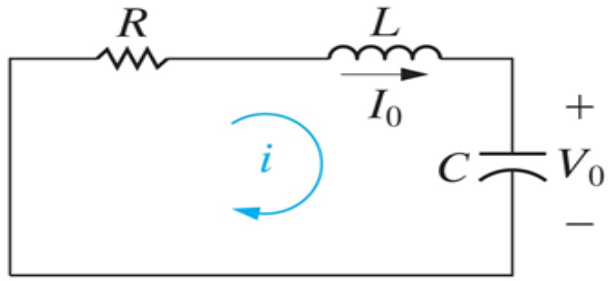
8.4 Natural and Step Responses of Series RLC Circuits
Summary of Natural Response of Series RLC Circuits

TABLE 8.4 Equations for analyzing the natural response of series *RLC* circuits

Characteristic equation	$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{R}{2L} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$
	$i(0^+) = A_1 + A_2 = I_0$
	$\frac{di(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{L}(-RI_0 - V_0)$
$\alpha^2 < \omega_0^2$: underdamped	$i(t) = B_1 e^{-\alpha t}\cos \omega_d t + B_2 e^{-\alpha t}\sin \omega_d t, \quad t \geq 0$
	$i(0^+) = B_1 = I_0$
	$\frac{di(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{L}(-RI_0 - V_0)$
$\alpha^2 = \omega_0^2$: critically damped	$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, \quad t \geq 0$
	$i(0^+) = D_2 = I_0$
	$\frac{di(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{L}(-RI_0 - V_0)$

Describing equation

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$



8.4 Natural and Step Responses of Series RLC Circuits
Natural Response of a Series RLC Circuit

1. Determine the initial capacitor voltage (V_0) and inductor current (I_0) from the circuit.
2. Determine the values of α and ω_0 using the equations in Table 8.4.
3. If $\propto^2 > \omega_0^2$, the response is overdamped, $\boldsymbol{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$, $t \geq 0$;

If $\propto^2 < \omega_0^2$, the response is underdamped, $\boldsymbol{i(t) = B_1 e^{-\alpha t} cos\omega_d t + B_2 e^{-\alpha t} sin\omega_d t}$, $t \geq 0$;

If $\propto^2 = \omega_0^2$, the response is critically damped, $\boldsymbol{i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$, $t \geq 0$;

4. If the response is overdamped, calculate s_1 and s_2 using the equations in table 8.4.

If the response is underdamped calculate ω_d using the equation in table 8.4.

5. If the response is overdamped, calculate A_1 and A_2 by simultaneously solving the equations in table 8.4.

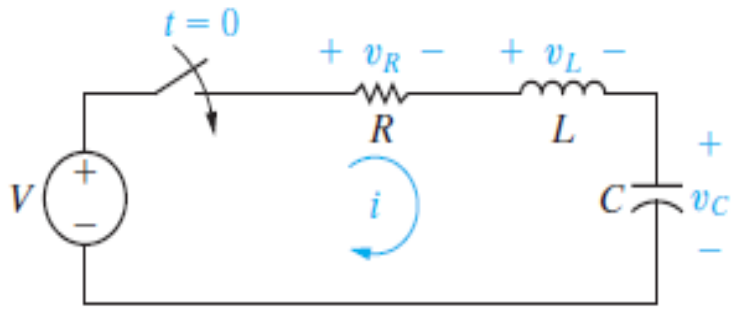
If the response is underdamped, calculate B_1 and B_2 by simultaneously solving the equations in table 8.4.

If the response is critically damped, calculate D_1 and D_2 by simultaneously solving the equations in table 8.4.

6. Write the equation for **i(t)** from Step 3 using the results of Steps 4 and 5; find any desired component voltages.

8.4 Natural and Step Responses of Series RLC Circuits

- The Step Response problem – find $v_c(t)$ for t ≥ 0. Note that there may or may not be initial energy stored in the inductor and capacitor!



- Apply KVL to the circuit.
- $V = Ri + L\frac{di}{dt} + v_c$

- Current is related to capacitor voltage by
- $i = C\frac{dv_c}{dt}, \Rightarrow \frac{di}{dt} = C\frac{d^2v_c}{dt^2}$, substitute into the above KVL equation.

- $V = RC\frac{dv_c}{dt} + LC\frac{d^2v_c}{dt^2} + v_c$, divide through by L$C$

- $\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_c}{LC} = \frac{V}{LC}$ (8.34)

- **same characteristics equation as the describing equation for its natural response counterpart!**

8.4 Natural and Step Responses of Series RLC Circuits
Summary of Step Response of Series RLC Circuits

- Describing equation:
  - $\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$

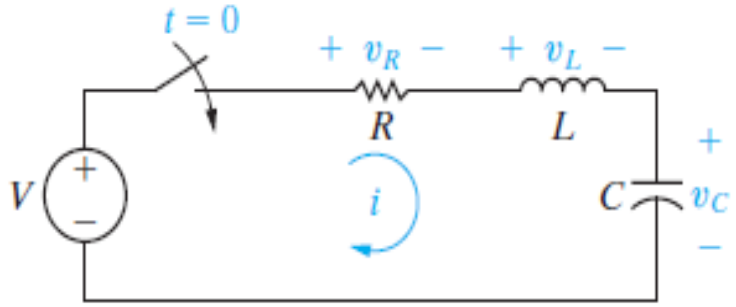


TABLE 8.5 Equations for analyzing the step response of series *RLC* circuits

Characteristic equation	$s^2 + \frac{R}{L}s + \frac{1}{LC} = \frac{V}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{R}{2L} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v_C(t) = V_f + A_1'e^{s_1t} + A_2'e^{s_2t}, t \geq 0$
	$v_C(0^+) = V_f + A_1' + A_2' = V_0$
	$\frac{dv_C(0^+)}{dt} = s_1A_1' + s_2A_2' = \frac{I_0}{C}$
$\alpha^2 < \omega_0^2$: underdamped	$v_C(t) = V_f + B_1'e^{-\alpha t}\cos\omega_d t + B_2'e^{-\alpha t}\sin\omega_d t, t \geq 0$
	$v_C(0^+) = V_f + B_1' = V_0$
	$\frac{dv_C(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = \frac{I_0}{C}$
$\alpha^2 = \omega_0^2$: critically damped	$v_C(t) = V_f + D_1'te^{-\alpha t} + D_2'e^{-\alpha t}, t \geq 0$
	$v_C(0^+) = V_f + D_2' = V_0$
	$\frac{dv_C(0^+)}{dt} = D_1' - \alpha D_2' = \frac{I_0}{C}$

8.4 Natural and Step Responses of Series RLC Circuits
Step Response of a Series RLC Circuit

1. Determine the initial capacitor voltage (V_0), the initial inductor current (I_0), and the final capacitor voltage (V_f) from the circuit.
2. Determine the values of α and ω_0 using the equations in Table 8.5.
3. If $\propto^2 > \omega_0^2$, the response is overdamped, $\boldsymbol{v_c(t) = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}}$, $t \geq 0^+$;

If $\propto^2 < \omega_0^2$, the response is underdamped, $\boldsymbol{v_c\,(t) = V_f + B_1' e^{-\alpha t} cos\omega_d t + B_2' e^{-\alpha t} sin\omega_d t}$, $t \geq 0^+$;

If $\propto^2 = \omega_0^2$, the response is critically damped, $\boldsymbol{v_c\,(t) = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}}$, $t \geq 0^+$;

4. If the response is overdamped, calculate s_1 and s_2 using the equations in table 8.5.

If the response is underdamped calculate ω_d using the equation in table 8.5.

5. If the response is overdamped, calculate $A_1'$and A_2' by simultaneously solving the equations in table 8.5.

If the response is underdamped, calculate B_1' and B_2' by simultaneously solving the equations in table 8.5.

If the response is critically damped, calculate $D_1'$and D_2' by simultaneously solving the equations in table 8.5.

6. Write the equation for $\boldsymbol{v_c}(t)$ from Step 3 using the results of Steps 4 and 5; find the capacitor current and any desired component voltages.

Important Note!

- What we have covered so far in second-order circuits is the derivation of the describing differential equation and forms of the solutions for standard parallel and series RLC circuits.

- When the circuit configuration is different from the standard form, you have to setup the describing differential equation using KVL and/or KCL, and solve the differential equation as usual using initial states of the circuit. This is one of the reasons why you should aim for understanding rather than memorizing the formula.

General Approach to Solving Second-Order DC Circuits

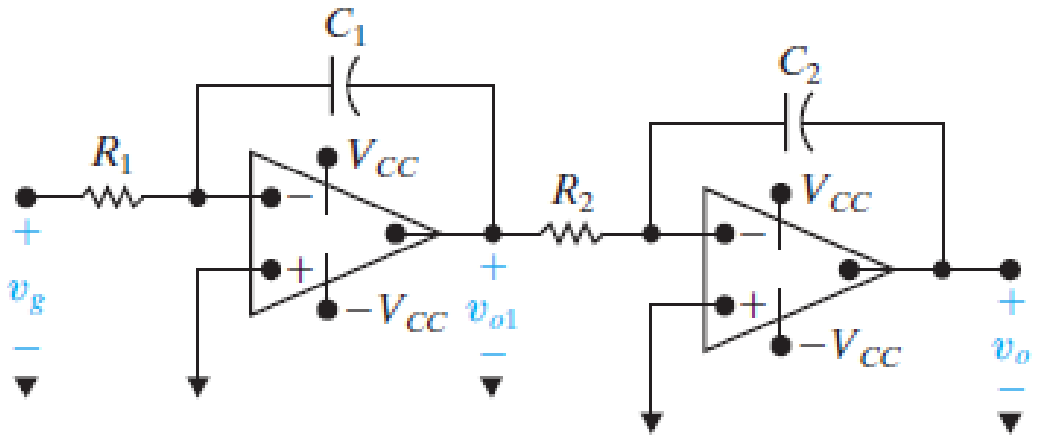
- There are many other second-order circuits (including second-order op amp circuits) apart from the standard parallel and series RLC circuit that we have covered so far. Given any second-order circuit, we determine its response, $x(t)$ (which may be voltage or current), as follows:

1. Determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$
2. Apply KCL and KVL to get equations describing the circuit and manipulate the equation until you get second-order differential equation with variable, $x(t)$, in the form: $\frac{d^2x(t)}{dt^2} + a\frac{dx(t)}{dt} + bx(t) = c$, where a, b, c are constants which are dependent on the parameters of the circuit.
3. Solve the differential equation as usual. Note that the particular solution is the same as $x(\infty)$. Apply initial conditions, $x(0)$ and $dx(0)/dt$, to the complete solution to determine the constants associated with the homogeneous solution (which is the natural response)
4. Check: use the initial, $x(0)$, and final values, $x(\infty)$, determined in (1) above to check your solution for accuracy

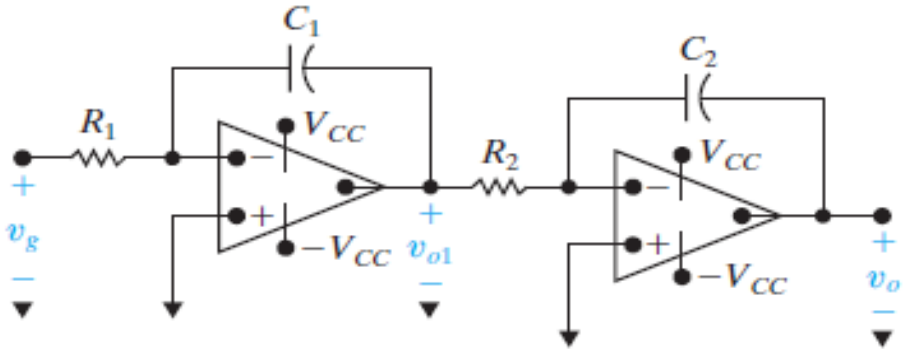
8.5 A Circuit with Two Integrating Amplifiers

- We often build second-order circuits with op amps rather than using inductors as inductors are difficult to integrate onto a chip and therefore do not lend themselves well to miniaturized circuits.
- The simplest way to do this would be to cascade two first-order op amp circuits

- In a cascade connection, the output of one subcircuit is the input to the second subcircuit.



8.5 A Circuit with Two Integrating Amplifiers

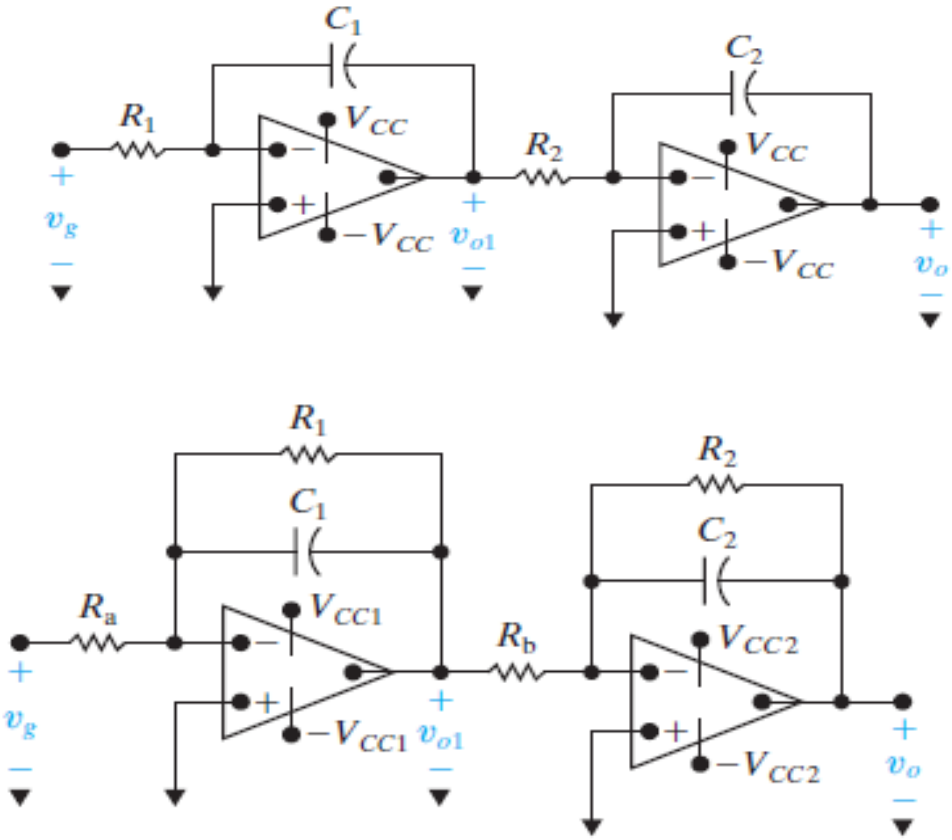


- Derive the differential equation that defines the relationship between v_o and v_g assuming that the op amps are ideal. Sum the currents at the inverting input terminal of the first integrator:
  - $\frac{0-v_g}{R_1} + C_1 \frac{d}{dt}(0 - v_{o1}) = 0$
- Simplifying and rearranging,
  - $\frac{dv_{o1}}{dt} = -\frac{1}{R_1 C_1} v_g$ (8.38)
- From the describing equation for the right integrating amplifier,
  - $\frac{0-v_{o1}}{R_2} + C_2 \frac{d}{dt}(0 - v_o) = 0$
  - $\frac{dv_o}{dt} = -\frac{1}{R_2 C_2} v_{o1}$
- Differentiating both sides
  - $\frac{d^2 v_o}{dt^2} = -\frac{1}{R_2 C_2} \frac{dv_{o1}}{dt}$
- Substituting (8.38), we get, $\frac{d^2 v_o}{dt^2} = \frac{1}{R_1 C_1} \frac{1}{R_2 C_2} v_g$ (8.39)

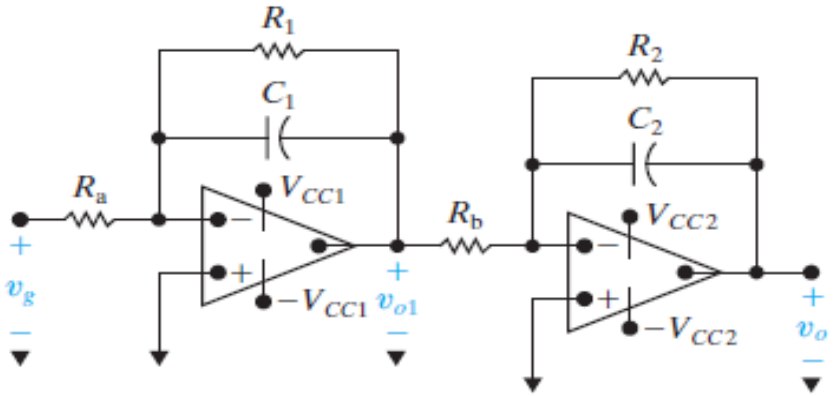
- If v_g is a constant input, that is DC, what kind of response function do we get at the output?
  - $s^2 = 0$; $s_{1,2} = 0$ twice

8.5 A Circuit with Two Integrating Amplifiers

- Both of the integrating amplifiers can saturate due to charge accumulating on the feedback capacitors. To overcome this problem, resistors are added in the feedback paths to get the following circuit:



8.5 A Circuit with Two Integrating Amplifiers



Derive the second-order differential equation that relates v_{o1} to v_g, by summing the currents at the inverting input node of the 1st integrator

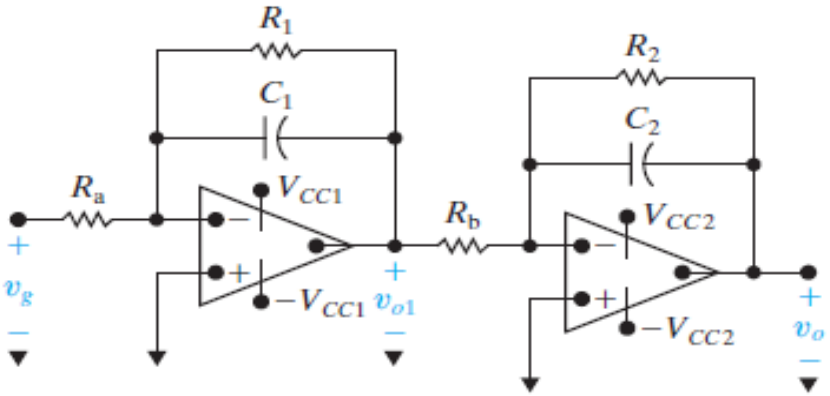
$$\frac{0 - v_g}{R_a} + \frac{0 - v_{o1}}{R_1} + C_1 \frac{d}{dt}(0 - v_{o1}) = 0$$

Simplifying and rearranging,

$$\frac{dv_{01}}{dt} + \frac{1}{R_1 C_1} v_{01} = \frac{-v_g}{R_a C_1} \qquad \text{let } \tau_1 = R_1 C_1$$

$$\frac{dv_{01}}{dt} + \frac{v_{01}}{\tau_1} = \frac{-v_g}{R_a C_1} \qquad (8.40)$$

8.5 A Circuit with Two Integrating Amplifiers



Next sum the currents at the inverting input node of the 2nd integrator

$$\frac{0 - v_{o1}}{R_b} + \frac{0 - v_o}{R_2} + C_2 \frac{d}{dt}(0 - v_o) = 0$$

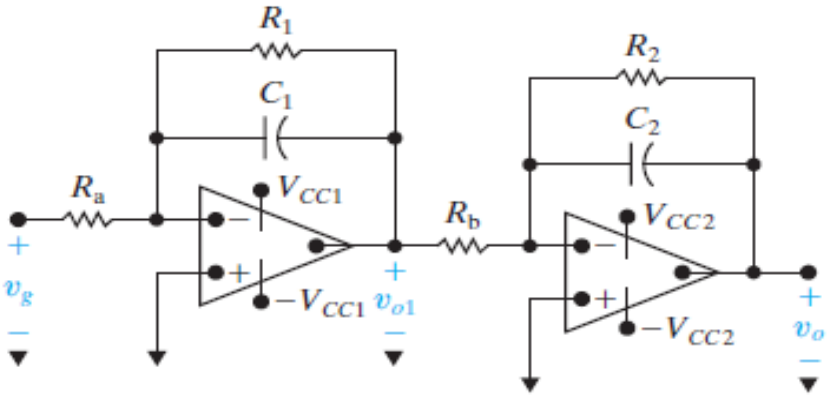
Simplifying and rearranging,

$$\frac{dv_0}{dt} + \frac{v_0}{\tau_2} = \frac{-v_{o1}}{R_b C_2} \qquad \text{where } \tau_2 = R_2 C_2 \qquad (8.41)$$

Differentiating both sides,

$$\frac{d^2 v_0}{dt^2} + \frac{1}{\tau_2}\frac{dv_0}{dt} = -\frac{1}{R_b C_2}\frac{dv_{01}}{dt}$$

8.5 A Circuit with Two Integrating Amplifiers



- Substituting the previous terms, yields this second-order differential equation:
- $$\frac{d^2v_o}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dv_o}{dt} + \left(\frac{1}{\tau_1\tau_2}\right)v_o = \frac{v_g}{R_aC_1R_bC_2} \qquad (8.42)$$
- Where $\tau_1 = R_1C_1$ and $\tau_2 = R_2C_2$

- The characteristic equation is $s^2 + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)s + \left(\frac{1}{\tau_1\tau_2}\right) = 0$
- The characteristic equation has real roots: $s_1 = -1/\tau_1$ and $s_2 = -1/\tau_2$ (only overdamped response form)

Complex Numbers (Math Review)

- Some math problems lead to determining the square root of a negative number. The following symbol is used to symbolize this square root.

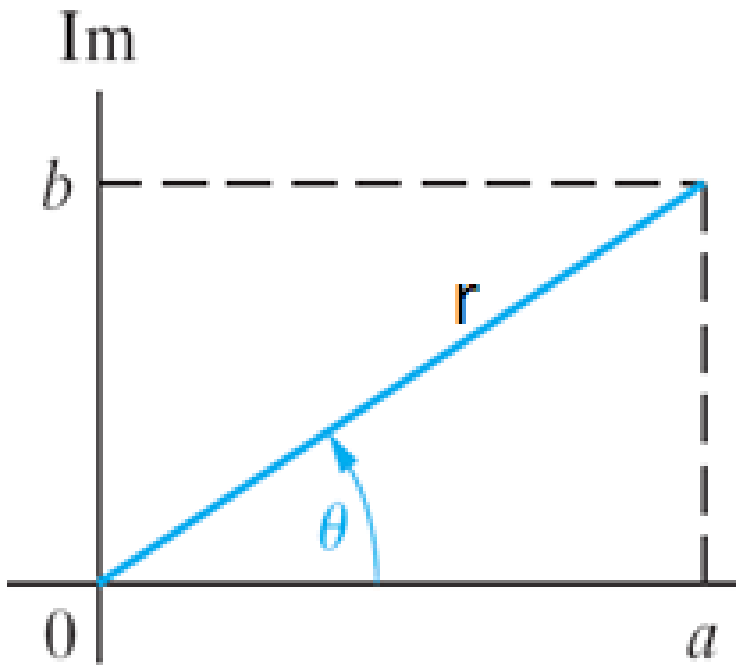
- $\sqrt{-1} = j$, or i

- In electrical engineering, we prefer the symbol j over i to avoid ambiguity with current symbols.

- j is regarded as the basic unit of the imaginary number since it is not an element of the set of real numbers

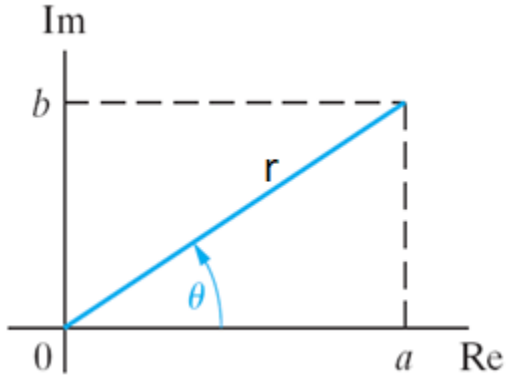
Complex Numbers (Math Review): Rectangular and Polar Representation

- **Rectangular Form:**
- Complex numbers are a set of numbers that can have both real and imaginary components, and can be represented as follows:
- $z = a + jb$ where a is the real component and b is the imaginary component



Complex Numbers (Math Review): Rectangular and Polar Representation

Polar Form:



- By Pythagorean theorem, $r = \sqrt{a^2 + b^2}$
- Applying some trig formulas to the figure on the right gives:
- $z = a + jb = rcos(\theta) + j\, rsin(\theta)$
- Euler's identity, $e^{j\theta} = cos(\theta) + jsin(\theta)$
- $z = a + jb = rcos(\theta) + jrsin(\theta) = re^{j\theta}$
- Where r is the magnitude and θ is the angle of the complex number given by

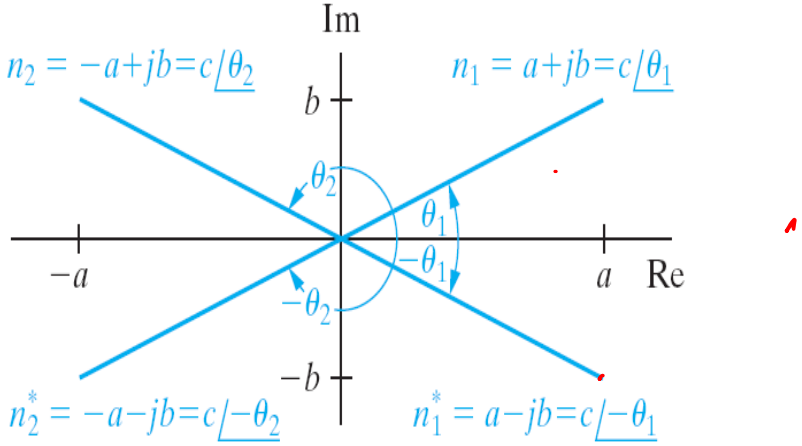
$$\theta = \tan^{-1}\frac{b}{a}$$

- $rcos(\theta) + jrsin(\theta)$ and $re^{j\theta}$ (also written as $r\angle\theta$) are considered the polar form of representing complex numbers. $re^{j\theta}$ is also called the exponential form.

Complex Numbers (Math Review): Conjugate of a Complex Number

- The conjugate of a complex number is formed by reversing the sign of the imaginary component if the complex number is given in rectangular form or reversing the sign of the angle if the complex number is given in polar form.

- In the figure, n_2^* is the complex conjugate of n_2 and vice versa.
- And n_1^* is the complex conjugate of n_1 and vice versa.



Complex Numbers (Math Review): Arithmetic Operations

- **Addition and Subtraction**
- Addition and subtraction of complex numbers are done in rectangular form. If the complex numbers to be added or subtracted are given in polar form, they should be converted to rectangular form before adding or subtracting.

- Given two complex number $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, addition and subtraction is carried out as follows:

- $z_1 + z_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$
- $z_1 - z_2 = (a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$

Complex Numbers (Math Review): Arithmetic Operations

- **Multiplication and Division**

- Multiplication and division is best carried out in polar form.

- Given two complex numbers $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, multiplication and division are carried out as follows

- $z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$

Complex Numbers (Math Review): Arithmetic Operations

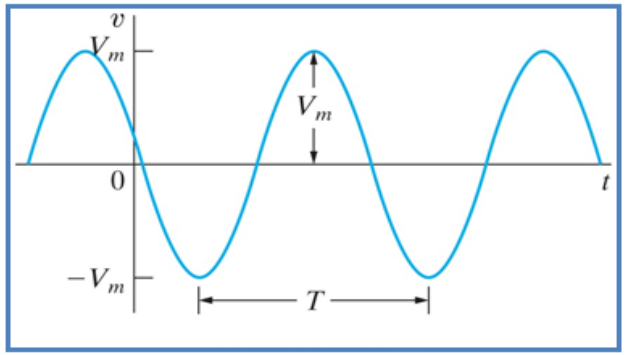
- **Multiplication and Division**
- In rectangular form.
- Given two complex numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, multiplication and division are carried out as follows

- $z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) = a_1 a_2 + j(a_1 b_2 + a_2 b_1) + j^2 b_1 b_2,$
- and $j^2 = -1$
- $z_1 z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$

- $\frac{z_1}{z_2} = \frac{(a_1+jb_1)}{(a_2+jb_2)} = \frac{(a_1+jb_1)(a_2-jb_2)}{(a_2+jb_2)(a_2-jb_2)} =$
- $\frac{(a_1a_2+b_1b_2)+j(-a_1b_2+a_2b_1)}{a_2^2+j(a_2b_2-a_2b_2)-j^2b_2^2} = \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + j\frac{(-a_1b_2+a_2b_1)}{a_2^2+b_2^2}$

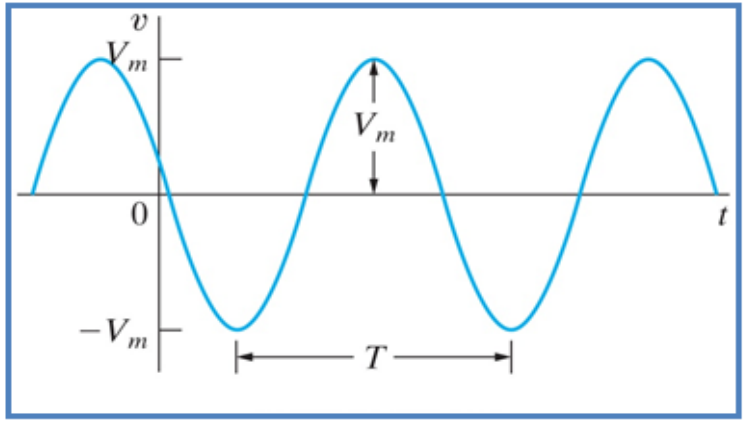
9.1 Sinusodal Source

- So far, we have focused on circuits with constant sources (DC sources).
- In the remaining chapters, we look at the steady-state analysis methods of circuits with sinusoidal sources. A sinusoid is a signal that has the form of a sine or cosine function
- Phasor transform is a technique that simplifies the analysis of such circuits.
- Fortunately, the previous circuit analysis techniques that we have covered are also valid in the phasor domain



9.1 Sinusodal Source

- $v(t) = V_m \cos(\omega t + \phi)$ V (9.1)
- V_m : maximum amplitude or magnitude – Since v(t) is a sinusoidal voltage, $\pm V_m$ bounds the amplitude.
- v(t) is a periodic function, since it repeats at regular intervals
- A periodic function is characterized by the time required for the function to pass through all of its possible values. This time is the period of the function
  - $T = 1/f$: period [s] f: frequency [Hz] (9.2)
- $\omega = 2\pi f = 2\,{}^{\pi}/_{T}$: angular frequency, [rad/s] (9.3)
- ϕ : phase angle
- Note ωt and ϕ must carry the same units because the argument of the sinusoidal function is $(\omega t + \phi)$



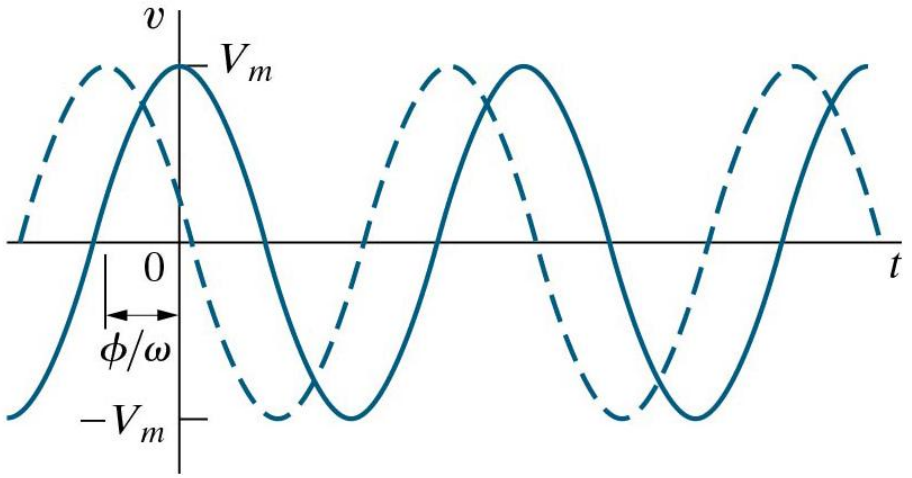
- Conversion from radians to degrees

(number of degrees) = $\frac{180}{\pi}$ (number of radians)

- $180^o = \pi$ radians
- $360^o = 2\pi$ radians

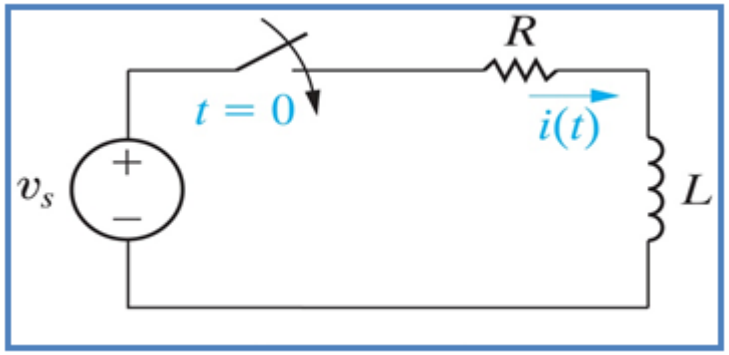
9.1 Sinusodal Source

- $v(t) = V_m \cos(\omega t + \phi)$ V
- When compared with a sinusoidal function with $\phi = 0$, a sinusoidal function with a positive ϕ is shifted to the left, while a sinusoidal function with a negative ϕ is shifted to the right.
- Another important characteristic of the sinusoidal voltage (or current) is the RMS value.



- $V_{rms} = \sqrt{\left(\frac{1}{T}\int_{t_o}^{t_o+T} V_m^2 cos^2(\omega t + \phi)\, dt\right)}$ **(9.4)**
- $V_{rms} = \frac{V_m}{\sqrt{2}}$ **(9.5)**
- **We can completely describe a sinusoidal signal if we know its frequency, phase angle, and amplitude.**

9.2 The Sinusoidal Response



- Suppose $v_s(t) = V_m \cos(\omega t + \phi)$ V in the circuit

Applying KVL to the circuit in the figure gives the ordinary differential equation for *i(t)* for $t \geq 0$ as, assuming the circuit's initial current is zero.

$$v_s(t) - Ri(t) - L\frac{di(t)}{dt} = 0$$

$$L\frac{di(t)}{dt} + Ri(t) = V_m \cos(\omega t + \phi) \qquad (9.6)$$

The solution of this equation for *i(t)* is

$$i(t) = \frac{-V_m}{\sqrt{R^2+(\omega L)^2}}\cos(\phi - \theta)e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2+(\omega L)^2}}\cos(\omega t + \phi - \theta) \qquad (9.7)$$

where $\theta = \tan^{-1}(\omega L/R)$

9.2 The Sinusoidal Response

- Consider the equation for the current in the RL circuit with the sinusoidal source:
- $i(t) = \frac{-V_m}{\sqrt{R^2+(\omega L)^2}}\cos(\phi-\theta)e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2+(\omega L)^2}}\cos(\omega t+\phi-\theta)$

$$\text{where} \quad \theta = \tan^{-1}(\omega L/R)$$

- Which term(s) go to 0 as t $\rightarrow \infty$?

- The first term is the transient component – it decays to 0 as *t* goes to infinity
- The second term is the steady-state component – it persists for all time greater than 0. This term
  - Is a cosine function, like the circuit's source
  - Has the same frequency as the input voltage source
  - Has a different magnitude and phase angle compared to the input voltage

9.2 The Sinusoidal Response

- The circuit analysis techniques we are studying will allow us to calculate the sinusoidal steady-state response of the circuit – that is, the circuit's response to a sinusoidal input once the transient response has effectively decayed to 0.
- This is also called the AC Steady-State (ACSS) response.

9.3 The Phasor

- A phasor is a complex number in polar form, with a magnitude and phase angle derived from a sinusoid using the phasor transform.

- Euler's identity: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

  $\Rightarrow \quad \cos\theta = \mathrm{Re}\{e^{j\theta}\} \qquad \sin\theta = Imag\{e^{j\theta}\}$

- Using the sinusoidal voltage function in (9.1)
- $v(t) = V_m \cos(\omega t + \phi)$

$$= V_m \mathrm{Re}\{e^{j(\omega t+\phi)}\}$$
$$= V_m \mathrm{Re}\{e^{j\omega t} e^{j\phi}\}$$
$$= \mathrm{Re}\{V_m e^{j\omega t} e^{j\phi}\}$$

9.3 The Phasor

Phasor Transform:

- Extracts a sinusoid's magnitude and phase angle
- Transforms a function of time into a function of frequency
- $\mathrm{V} = \mathrm{P}\{V_m \cos(\omega t + \phi)\} = V_m e^{j\phi} = V_m \angle \phi$ (9.8)

Inverse Phasor Transform:

- Turns a phasor back into a sinusoid, if you are given the frequency
- $v(t) = \mathrm{P}^{-1}\{V_m e^{j\phi}\} = \mathrm{P}^{-1}\{V_m \angle \phi\} = V_m \cos(\omega t + \phi)$ (9.9)

9.3 The Phasor

Translating a sine expression to a cosine expression

- We can translate a sine function to a cosine function by subtracting 90° ($\frac{\pi}{2}\ rad$) from the argument of the sine function
  - $\sin(\omega t + \theta) = cos(\omega t + \theta - 90^o)$

- Verification involves using the identity
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- Let $\alpha = \omega t + \theta$ and $\beta = 90^o$
- $cos(\omega t + \theta - 90^o) = cos(\omega t + \theta)\cos(90^o) + \sin(\omega t + \theta)\sin(90^o)$
- Since $\cos(90^o) = 0$ and $\sin(90^o) = 1$, we have
- $cos(\omega t + \theta - 90^o) = \sin(\omega t + \theta)$

- Hence for example,
- $\sin(\omega t + 30^o) = cos(\omega t + 30^o - 90^o) = cos(\omega t - 60^o)$

9.3 The Phasor

- Adding or subtracting sinusoids in the time domain is hard (you need trig identities!) But if you apply phasor-transform to the sinusoids, it is easy to combine them.
- For example,
- $i(t) = [5\cos(300t + 36.87°) + 10\cos(300t - 53.13°)]\ \text{A}$

$$= 10 - j5$$

- $\mathrm{I} = 5\angle 36.87° + 10\angle -53.13° = 11.18\angle -26.57°\ \text{A}$

$$\therefore \quad i(t) = \mathrm{P}^{-1}\{\mathrm{I}\} = 11.18\cos(300t - 26.57°)\ \text{A}$$

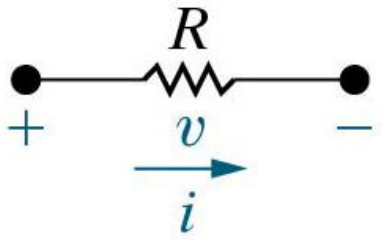
$5(\cos 36.87 + j\sin 36.87°)$

$5(0.8 + j0.6) = 4 + j3$

$6 - j8$

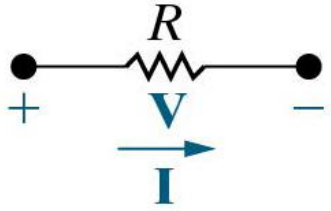
9.4 The Passive Circuit Elements in the Frequency Domain

Resistors (Time Domain)



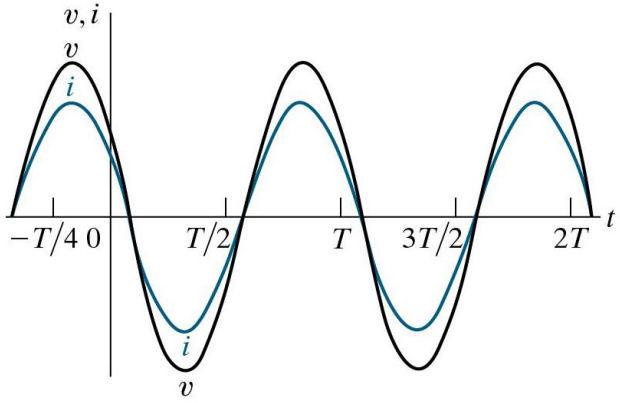
$$i(t) = I_m \cos(\omega t + \theta_i)$$
$$\therefore \quad v(t) = Ri(t) = RI_m \cos(\omega t + \theta_i)$$

Resistors (Phasor/Frequency Domain)



$$\mathrm{I} = I_m \angle \theta_i$$
$$\therefore \quad \mathrm{V} = R\mathrm{I} = RI_m \angle \theta_i \qquad \textbf{(9.10)}$$

Note that the voltage and current phasors for a resistor have the same phase angle – therefore we say that the voltage and current in a resistor are “in phase”.



9.4 The Passive Circuit Elements in the Frequency Domain

Inductors (Time Domain)

$$i(t) = I_m \cos(\omega t + \theta_i)$$

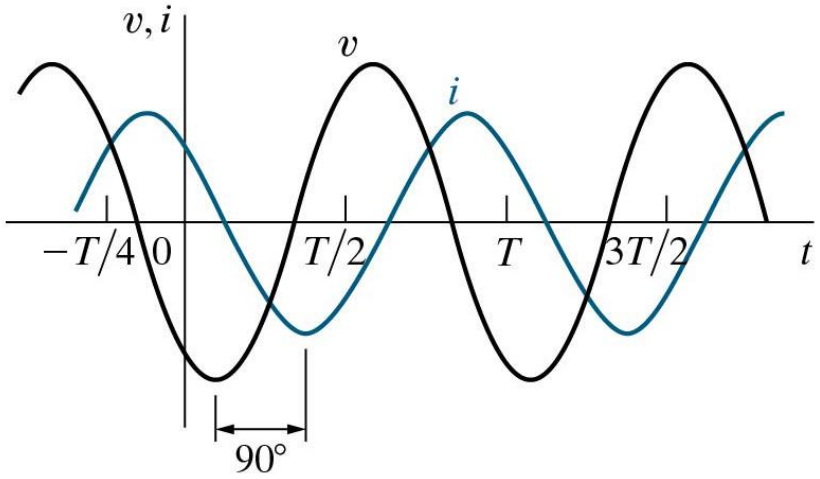
$$\therefore \quad v(t) = L\, di(t)/dt$$

$$= -\omega L I_m \sin(\omega t + \theta_i)$$

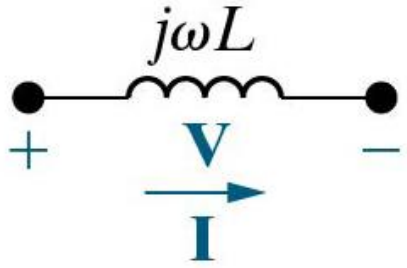
$$= -\omega L I_m \cos(\omega t + \theta_i - 90°)$$

$$= \omega L I_m \cos(\omega t + \theta_i + 90°)$$

In an inductor, the voltage leads the current by 90°



Inductors (Phasor/Frequency Domain)



$$\mathrm{I} = I_m \angle \theta$$

$$\therefore \quad \mathrm{V} = -\omega L I_m e^{j(\theta_i - 90°)}$$

$$= -\omega L I_m e^{j\theta_i} e^{-j90°}$$

$$= -\omega L I_m e^{j\theta_i}(-j)$$

$$= j\omega L I_m e^{j\theta_i}$$

$$= j\omega L\, I_m < \theta_i$$

$$= j\omega L \mathrm{I} \qquad \textbf{(9.11)}$$

Also can be written as,

$$V = (\omega L < 90^o)\; I_m < \theta_i$$

$$= \omega L\; I_m < (\theta_i + 90)^o$$

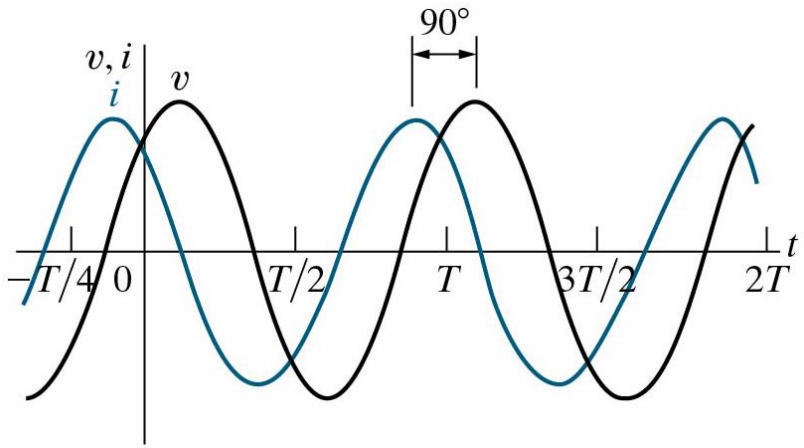
9.4 The Passive Circuit Elements in the Frequency Domain

Capacitors (Time Domain)

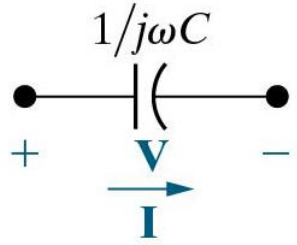
$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$\therefore \quad i(t) = C\, dv(t)/dt$$
$$= -\omega C V_m \sin(\omega t + \theta_v)$$
$$= -\omega C V_m \cos(\omega t + \theta_v - 90°)$$
$$i(t) = -\omega C V_m \cos(\omega t + \theta_v - 90°)$$

In a capacitor, the voltage lags the current by 90°



Capacitors (Phasor/Frequency Domain)



$$\mathrm{V} = V_m \angle \theta$$

$$\therefore \quad \mathrm{I} = -\omega C V_m e^{j(\theta_v - 90°)}$$
$$= -\omega C V_m e^{j\theta_v} e^{-j90°}$$
$$= -\omega C V_m e^{j\theta_v}(-j)$$
$$= j\omega C V_m e^{j\theta_v}$$
$$= j\omega C V_m < \theta_v$$
$$= j\omega C \mathrm{V}$$

$$V = \frac{1}{jwC} I \qquad \textbf{(9.12)}$$

$$V = (\frac{1}{\omega C} < -90^o)\ I_m\ < \theta_i$$
$$= \frac{I_m}{\omega C} < (\theta_i - 90)^o$$

9.4 The Passive Circuit Elements in the Frequency Domain

VI Comparison in Time and Phasor Domain

- In the time domain
  - Resistor: $v(t) = Ri(t)$
  - Inductor: $v(t) = L\frac{di}{dt}$
  - Capacitor: $i(t) = C\frac{dv}{dt}$

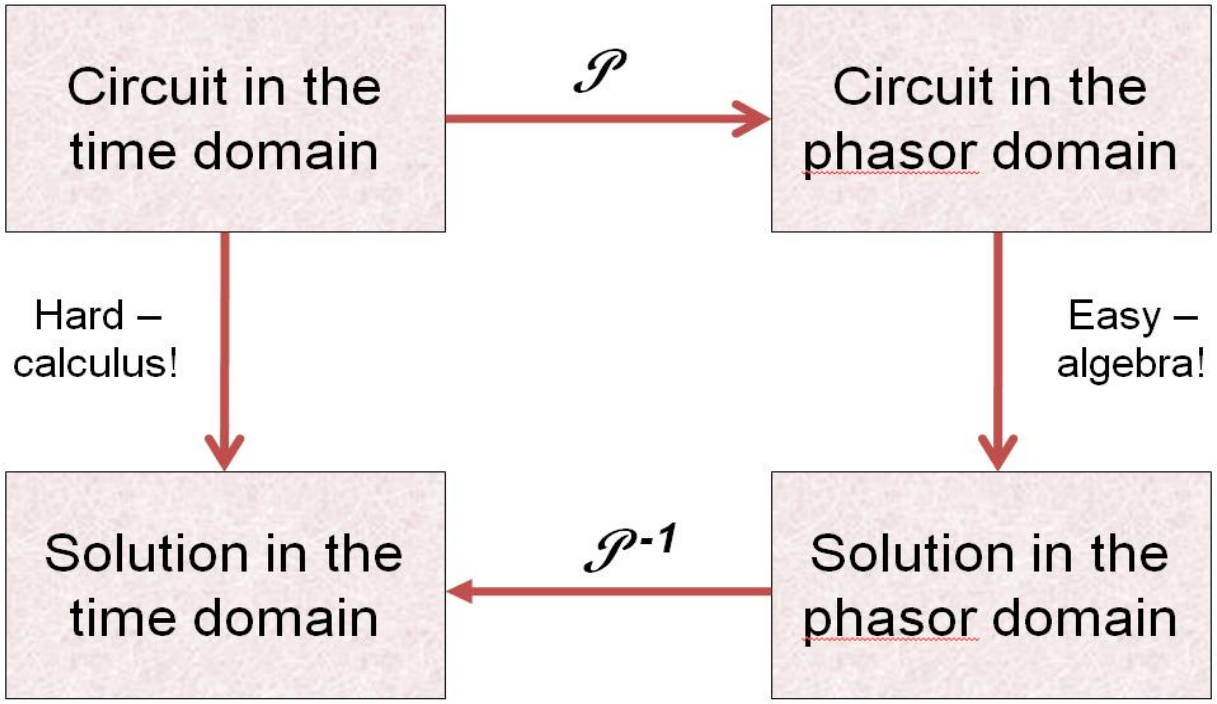
- In the phasor domain $V = ZI$ **(9.13)**
  - Z is impedance, defined as the ratio of V to I
  - Z has the units Ohms [Ω]
  - Resistor: $Z_R = R$
  - Inductor: $Z_L = j\omega L$
  - Capacitor: $Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$

9.4 The Passive Circuit Elements in the Frequency Domain

$$Z = R + jX$$

Circuit Element	Impedance	Reactance
Resistor	R	—
Inductor	$j\omega L$	ωL
Capacitor	$j(-1/\omega C)$	$-1/\omega C$

Motivation for Transforming to Phasor Domain



9.5 Kirchoff's Laws in the Frequency Domain

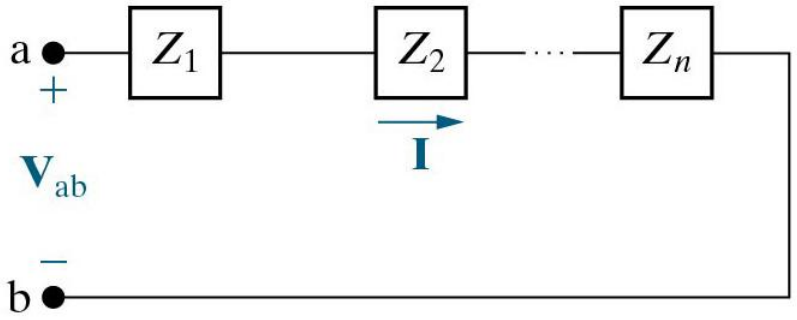
- In the time domain
  - Ohm's law (only for resistors): $v = Ri$
  - KVL (around a loop): $v_1 + v_2 + \ldots + v_n = 0$
  - KCL (at a node): $i_1 + i_2 + \ldots + i_n = 0$
  - These three laws lead to all other time-domain circuit analysis techniques

- In the phasor domain
  - Ohm's law (for R, L, C): $V = ZI$
  - KVL (around a loop): $V_1 + V_2 + \ldots + V_n = 0$ **(9.14)**
  - KCL (at a node): $I_1 + I_2 + \ldots + I_n = 0$ **(9.15)**
  - **These three laws mean we can use all time-domain circuit analysis techniques in the phasor domain!**

9.6 Series, Parallel, and Delta-to-Wye Simplifications

Combining Impedances in Series

- $Z_{ab} = \frac{V_{ab}}{I} = Z_1 + Z_2 + \cdots + Z_n$ **(9.16)**

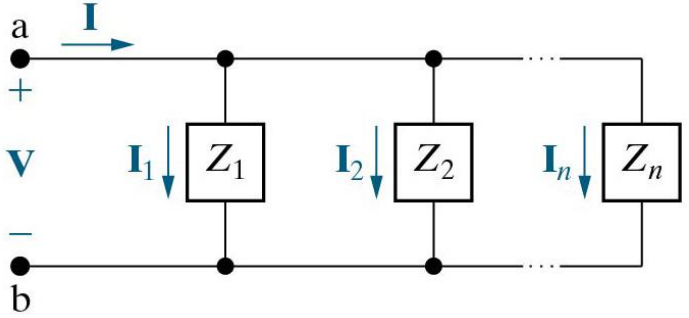


Voltage Division in the Frequency Domain

- $V_j = \frac{Z_j}{Z_{eq}} V_s$ **(9.17)**
- V_s is the voltage applied to a collection of series-connected impedances, V_j is the voltage across the impedance of Z_j, and Z_{eq} is the equivalent impedance of the series-connected impedances

9.6 Series, Parallel, and Delta-to-Wye Simplifications

- Combining Impedances in Parallel
  - $\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$ **(9.18)** $\quad Z_{ab} = \frac{Z_1 Z_2}{Z_1 + Z_2}$ **(9.19)**



- Current Division in the Frequency Domain
  - $I_j = \frac{Z_{eq}}{Z_j} I_s$ **(9.20)**
- I_s is the current supplied to a collection of parallel-connected impedances, I_j is the current in the branch containing the impedance Z_j and Z_{eq} is the equivalent impedance of the parallel-connected impedances

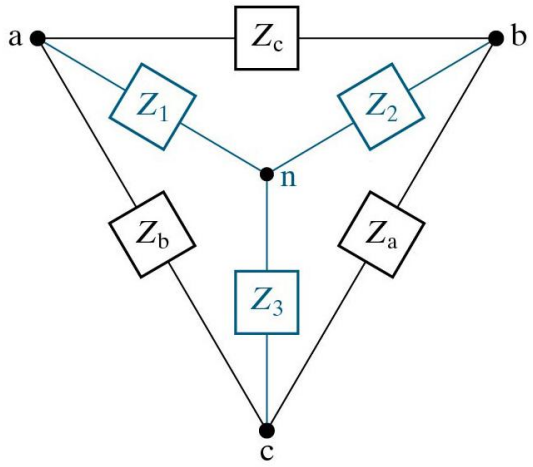
9.6 Series, Parallel, and Delta-to-Wye Simplifications

- Admittance is the reciprocal of impedance, and denoted Y.
  - $Y = \frac{1}{Z}$ = G + jB (siemens)
- Admittance is a complex number whose real part, G, is called **conductance** and whose imaginary part, B, is called **susceptance**. Conductance and susceptance are measured in siemens (S).
- Replacing impedances with admittances in (9.18) gives
  - $Y_{ab} = Y_1 + Y_2 + \cdots + Y_n$

9.6 Series, Parallel, and Delta-to-Wye Simplifications

Delta-Wye Transformations

- The Δ-Y transformation for resistive circuits, discussed in chapter 3 also applies to impedances.
- For the circuit shown,
  - Y impedances as functions of the Δ impedances
  - Δ impedances as functions of Y impedances



(9.21)-(9.23)

$$Z_1 = \frac{Z_b \cdot Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_a \cdot Z_c}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a \cdot Z_b}{Z_a + Z_b + Z_c}$$

(9.24)-(9.26)

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_1}$$

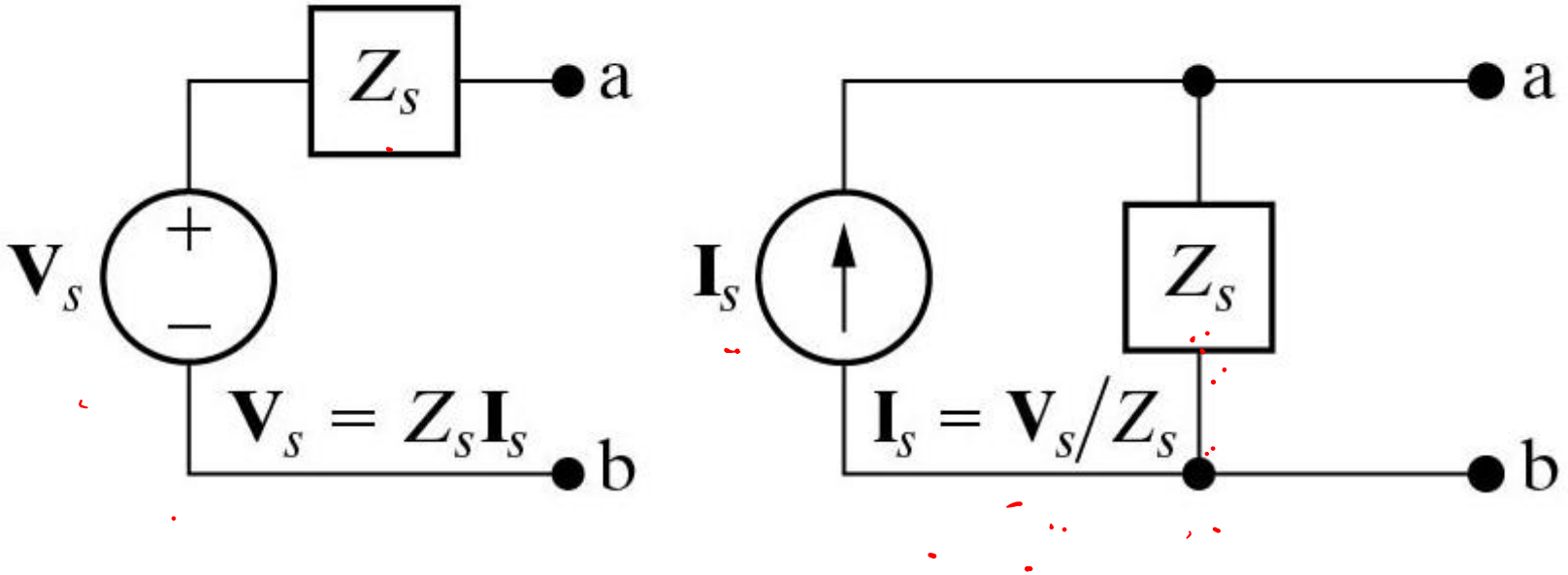
$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_3}$$

9.7 Source Transformations and Thevenin-Norton Equivalent Circuits

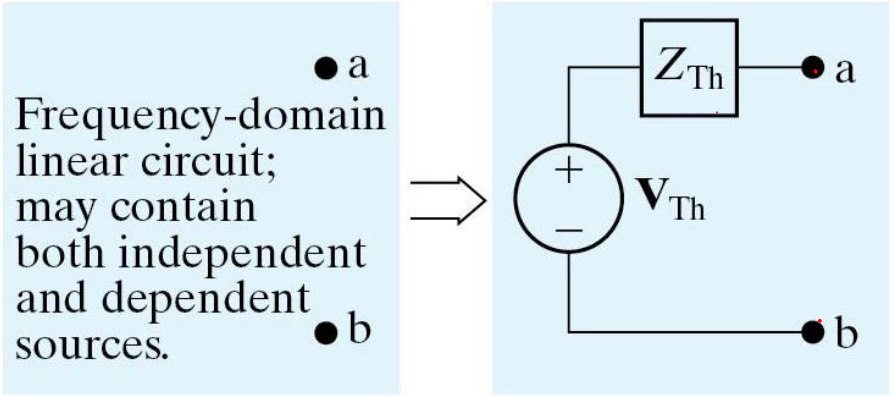
- The source transformations and Thevenin-Norton equivalent circuits introduced in chapter 4 are analytical techniques that also can be applied to frequency domain circuits

Source Transformation

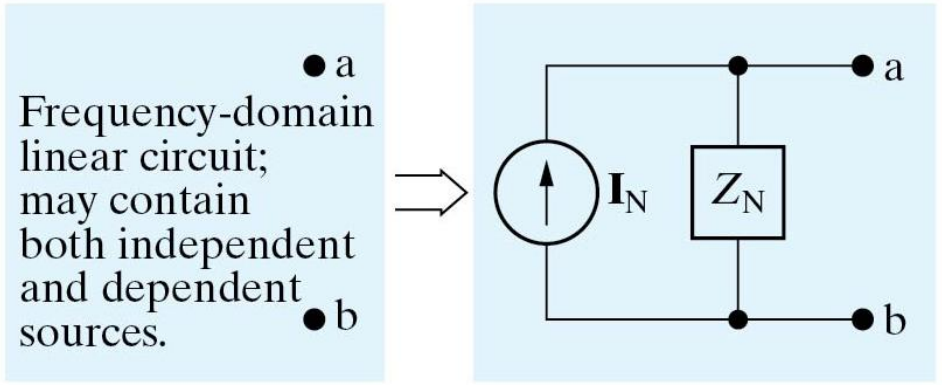


9.7 Source Transformations and Thevenin-Norton Equivalent Circuits

Thevenin Equivalent Circuit



Norton Equivalent Circuit



9.8 Node-Voltage Method

- **Step Phasor** – If the circuit is in time domain, it must be transformed to the appropriate frequency domain.

- **Follow the steps of the Node-Voltage Circuit Analysis Method (chapter 4) to find the values of the unknown voltage and current phasors of interest.**

- **Step 1** – Identify all essential nodes in the circuit (where three or more elements are connected).
- **Step 2** – Choose one essential node as a reference node (usually the one with the most elements connected). Label that node with a and treat the voltage at that point as zero.
- **Step 3** –
  - – Label unknown voltages at all other essential nodes: $V_1, V_2, \ldots$ etc.
  - – Identify currents entering/leaving essential nodes in terms of $V_1, V_2, \ldots$
  - Use KCL at each nonreference essential node to set up equations to solve for the unknown voltages.
  - -- If there are dependent sources, write a constraint equation for each one.
  - -- If there are supernodes, write a constraint equation for each one.
- **Step 4** – Solve equations to find the node-voltage values.
- **Step 5** – Solve the circuit using node voltages from Step 4 to find component currents, voltages, and power values.

- **Step Inverse Phasor --** Apply the inverse phasor transform to the voltage and current phasors to find the steady-state values of the corresponding voltages and currents in the time domain.

9.9 Mesh-Current Method

- **Step Phasor** – If the circuit is in time domain, it must be transformed to the appropriate frequency domain.

- **Follow the steps of the Mesh-Current Circuit Analysis Method (chapter 4) to find the values of the unknown voltage and current phasors of interest.**

- **Step 1** – Identify all meshes (loops that do not enclose any other loops) with curved directed arrows that follow the perimeter of each mesh
- **Step 2 --** Label a mesh current for each mesh.
- **Step 3** –
- -- Identify voltages across all resistors in each mesh and write in terms of mesh currents.
- Use KVL to write an equation for each mesh.
- -- If there are dependent sources, write a constraint equation for each one.
- -- If there are supermeshes, write a constraint equation for each one.
- **Step 4** – Solve the KVL equations to find the mesh current values
- **Step 5** – Solve the circuit using mesh currents from step 4 to find component currents, voltages, and power values.

- **Step Inverse Phasor --** Apply the inverse phasor transform to the voltage and current phasors to find the steady-state values of the corresponding voltages and currents in the time domain.